TRIVANDRUM

(Pages : 3)

E – 1706

Reg. No. : .....

Name : .....

Sixth Semester B.Sc. Degree Examination, April 2018
First Degree Programme Under CBCSS
Mathematics
Core
MM 1644 : ABSTRACT ALGEBRA – II
(2014 Admission Onwards)

Time : 3 Hours

Max. Marks : 80

SECTION – I

All the **first 10** questions are **compulsory**. **Each** carries **1** mark.

1. Let $\phi : \mathbb{R} \to \mathbb{R}^*$, where $\mathbb{R}$ is the additive group and $\mathbb{R}^*$ is the multiplicative group, be given by $\phi(x) = 2^x$. Check whether ϕ is a homomorphism.
2. Find the order of $(\mathbb{Z}_3 \times \mathbb{Z}_5) / (\{0\} \times \mathbb{Z}_5)$.
3. Find the characteristic of the ring $\mathbb{Z}_3 \times 3\mathbb{Z}$.
4. State whether true or false : "The direct product of two integral domains is again an integral domain".
5. How many homomorphisms are there of $\mathbb{Z}$ into $\mathbb{Z}_2$?
6. Compute $(-3, 5)\ (2, -4)$ in $\mathbb{Z}_4 \times \mathbb{Z}_{11}$.
7. Find all units in the ring $\mathbb{Q}$ of rational numbers.
8. Give an example of a ring having two elements a and b s ab = 0 but neither a nor b is zero.
9. Find the solution of the congruence $36x \equiv 15 \pmod{24}$, if it exists.
10. If $\phi : S_n \to \mathbb{Z}_2$ be defined by $\phi(\sigma) = \begin{cases} 0, \text{ if } \sigma \text{ is an even permutation} \\ 1, \text{ if } \sigma \text{ is an odd permutation} \end{cases}$ find ker ϕ.

P.T.O.

SECTION – II

Answer **any 8** questions from this Section. **Each** question carries **2** marks.

11. Find all units in the ring $M_2(\mathbb{Z}_2)$.

12. Let F be the ring of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ and let N be the subring of all functions f such that f(2) = 0. I_s N is an ideal in F ?

13. Prove that if $\phi : G \to G'$ is a group homomorphism then ker ϕ is a normal subgroup of G.

14. Let $\phi : G \to G'$ is a group homomorphism. Show that if $|G'|$ is finite, then $|\phi(G)|$ is finite and is a divisor of $|G'|$.

15. Let $\phi : G \to G'$ is a group homomorphism and let N be a normal subgroup of G. Show that ϕ [N] is a normal subgroup of ϕ [G].

16. Is the converse of Lagrange's theorem true ? Justify your answer.

17. Show that an intersection of ideals of a ring R is again an ideal of R.

18. Solve the congruence $22x \equiv 5 \pmod{15}$.

19. Describe all group homomorphisms of $\mathbb{Z}$ into $\mathbb{Z}$.

20. Find all solutions of the equation $x^2 + x - 6 = 0$ in the ring $\mathbb{Z}_{14}$.

21. Prove that factor group of a cyclic group is cyclic.

22. Compute the factor group $(\mathbb{Z}_4 \times \mathbb{Z}_6) / \langle (0, 1) \rangle$.

SECTION – III

Answer **any 6** questions from this Section. **Each** question carries **4** marks.

23. Prove that a group homomorphism $\phi : G \to G'$ is a one-one map if and only if ker $\phi = \{e\}$.

24. Show that all automorphisms of a group G form a group under function composition.

25. Show that cancellation laws hold in a ring R if and only if R has no divisors of 0.

26. An element a of ring R is idempotent if $a^2 = a$. Show that a division ring contains exactly two idempotent elements.

27. Show that the characteristic of an integral domain D must be either zero or a prime p.

28. Show that $a^2 - b^2 = (a + b)(a - b)$ for all a and b in a ring R if and only if R is commutative.

29. State and prove Euler's Theorem.

30. Show that the factor ring of a field is either the trivial ring of one element or is isomorphic to the field.

31. Let $R = \{a + b\sqrt{2}, a, b \in \mathbb{Z}\}$ and let R' consists of all 2×2 matrices of the form $\begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$, for $a, b \in \mathbb{Z}$. Show that $\phi : R \to R'$ where $\phi(a + b\sqrt{2}) = \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$ is a ring isomorphism.

SECTION – IV

Answer **any 2** questions from this Section. **Each** question carries **15** marks.

32. a) Let H be a subgroup of a group G. Then the left coset multiplication is well defined by the equation (aH) (bH) = (ab)H if and only if H is a normal subgroup of G.
    b) Show that the set of all $g \in G$ such that $i_g : G \to G$ is the identity inner automorphism i_e, is a normal subgroup of a group G.

33. Prove that
    a) Every field is an Integral Domain.
    b) Every finite integral domain is a field.
    c) If p is a prime, then $\mathbb{Z}_p$ is a field.

34. State and prove fundamental theorem of group homomorphism.

35. Let $\phi : G \to G'$ is a group homomorphism. Prove that
    a) If e is the identity element in G, then $\phi(e)$ is the identity element in G'.
    b) If $a \in G$ then $\phi(a^{-1}) = (\phi(a))^{-1}$.
    c) If H is a subgroup of G, then $\phi[H]$ is a subgroup of G'.
    d) If K' is a subgroup of G', then $\phi^{-1}[K']$ is a subgroup of G.